Simple models of strings of characters with infinite alphabet

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Abstract

We consider the family of Markov chains defined on the sequences of characters (strings, or words) with infinite alphabet. For some examples inspired by the models of high frequency trading we obtain conditions for ergodicity, transience and null-recurrence. In order to prove this we use the construction of Lyapunov functions techniques.

Model description

We consider a homogeneous Markov chain ξ with discrete time and countable number of states $E = \cup_{n=0}^{\infty} \mathbb{Z}_+^n$, in other words consider finite strings (sequence of symbols) $\alpha = a_1 a_2 \dots a_n$ where n is length of string and symbols $a_k \in \mathbb{Z}_+$. With probability $q(a,\emptyset)$ one removes the last symbol a, $\gamma a \longrightarrow \gamma$, with probability $q(a,b)$ the last symbol a one changes into a symbol b, $\gamma a \longrightarrow \gamma b$, and with probability $q(a,bc)$ the last symbol a is substituted into the pair bc, $\gamma a \longrightarrow \gamma bc$. Let τ_a be a first moment when the length of the string γa decrease in 1. Let $e_a = E\tau_a$ the following equations holds true.

$$e_a = 1 + \sum_b q(a,b)\, e_b + \sum_{b,c} q(a,bc)\,(e_b + e_c)$$

Or in vector form

$$\vec{e} = \vec{1} + M\vec{e}$$

Where M is the matrix with elements

$$m_{ab} = q(a,b) + \sum_c q(a,bc) + \sum_c q(a,cb)$$

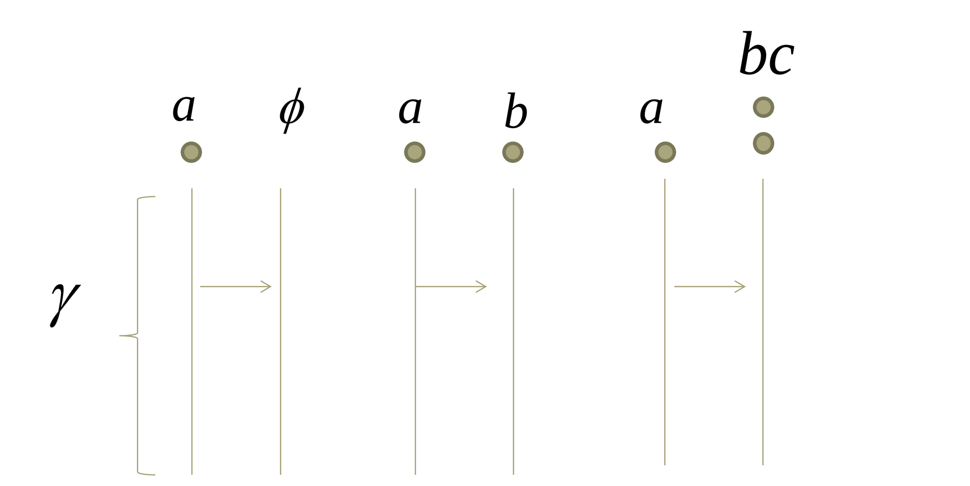

Fig1. possible transitions

Examples

Case 1

Case when $q(a,\emptyset) = 0$, let $q(a,a+1) = \lambda$, $q(a,a1) = \delta$ for all a, $q(a,a-1) = \mu$, for $a > 1$ and $q(1,\emptyset) = \mu$ where $\lambda + \delta + \mu = 1$. All other transition probabilities are 0s. Then the matrix M has the form: $m_{a,a+1} = \lambda$, $m_{11} = 2\delta$, $m_{aa} = \delta$ for $a > 1$, $m_{21} = \delta + \mu$, $m_{a,a-1} = \mu$ when $a > 2$, and $m_{a1} = \delta$ for $a > 2$.

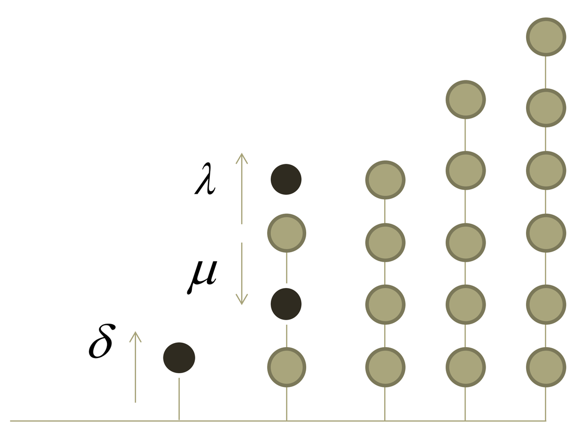

Fig.2 transitions in case 1

Lemma 1

- If $\lambda + \delta < \mu$ then the chain is ergodic.
- If $\lambda + \delta = \mu$ chain is null recurrence.
- If $\lambda + \delta > \mu$ then the chain is transient.

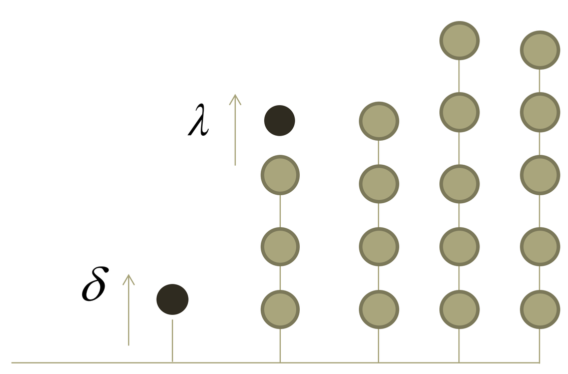

Fig.3 transitions in case 2

Case 2

Case when $q(a,\emptyset) = 0$, let $q(a,a+1) = \lambda$, $q(a,a1) = \delta$ for all a, where $\lambda + \delta = 1$.

All other transition probabilities are 0s. Then the matrix M has the form: $m_{a,a+1} = \lambda$, $m_{11} = 2\delta$, $m_{aa} = \delta$ for $a > 1$ and $m_{a1} = \delta$ for $a > 1$.

Lemma 2

- The chain is always transient.

Proof of Lemma 1 and 2

The Lyapunov function in these two cases will be

$f(\alpha) = \sum_{k=1}^{n} a_k$ Where $\alpha = a_1 a_2 \dots a_n$ and $\alpha \neq \emptyset$. Then an increment of the Lyapunov function will be: for case 1 is $\lambda + \delta - \mu$ and for case 2 is $\lambda + \delta$.

Hypothesis

It is known (see Vere-Jones), that for a matrix with non-negative elements and irreducible all power series $\sum_n m_{ab}^{(n)} z^n$ Converge and diverge simultaneously and all have common convergence radius R. For finite matrix $R = \lambda^{-1}$, where λ is the maximal eigenvalue.

Consider the minor principal $M_{(N)}$ of the matrix M. Let $R_{(N)}$ be radius of convergence of $M_{(N)}$.

Theorem 1 $R_{(N+1)} < R_{(N)}$ for all irreducible $M_{(N)}$ and $R_{(N)} \downarrow R$

There exists a matrix classification of matrices with non-negative elements analogous to the classification of stochastic matrices: R-transient, R-recurrent and R-ergodic.

Hypothesis

- If $R < 1$ - transience
- If $R > 1$ and matrix M is R-ergodic, then the chain is ergodic.
- If $R > 1$ and matrix M is R-transient, then the chain is transient.

References

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